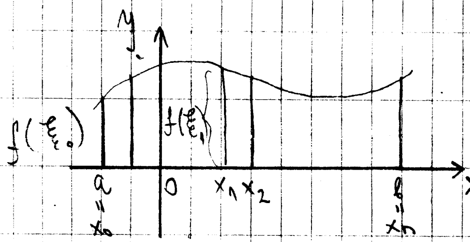
06.04.2010

ODREĐENI INTEGRAL

$$\int_a^b f(x)\,dx \overset{def.}{=} \lim_{n\to\infty} \sum_{i=0}^{n-1} f(\xi_i)\,\Delta x_i$$



$x_0 = a < x_1 < x_2 < \dots < x_{n-1} < x_n = b$

- ekvidistantna podjela:

$$\underbrace{x_1 - x_0}_{\Delta x_0} = \underbrace{x_2 - x_1}_{\Delta x_1} = \dots = \underbrace{x_n - x_{n-1}}_{\Delta x_{n-1}} = h$$

$b - a = n \cdot h \Rightarrow h = \frac{b-a}{n}$ (podjela ne mora uvjek biti ekvidistantna)

$\xi_i \in [x_i, x_{i+1}]$ $(i = 0, 1, 2, \dots, n-1)$

NEWTON-LEIBNIZ-ova FORMULA

$$F'(x) = f(x) \Rightarrow \int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

Osobine:

1. $\int_a^b f(x)\,dx = 0$
2. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
3. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^{b_1} f(x)\,dx + \int_{b_1}^{b_2} f(x)\,dx + \dots + \int_{b_{n-1}}^{b} f(x)\,dx$
4. $\int_a^b c \cdot f(x)\,dx = c \int_a^b f(x)\,dx$

5. $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

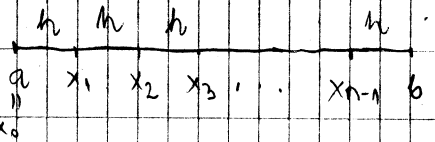
6. $f(x) \leq g(x)$ za $x \in [a, b]$

$\Rightarrow \int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$

7. $\int_{-a}^{a} f(x)\,dx = \begin{cases} 0, & f \text{ - neparna} \\ 2 \cdot \int_0^a f(x)\,dx, & f \text{ parna} \end{cases}$

(1) Izračunati po definiciji:

a) $\int_a^b \frac{dx}{x^2}, \quad 0 < a < b$



$n \cdot h = b - a \Rightarrow h = \frac{b-a}{n}$

$x_1 = x_0 + h = a + h$

$x_2 = x_0 + 2h = a + 2h$

$\dots$

$x_k = x_0 + kh = a + kh \quad (k = 0, 1, \dots, n)$

$x_n = b$

$\xi_i = \sqrt{x_i \cdot x_{i+1}} \in [x_i, x_{i+1}] \quad (i = 0, 1, \dots, n-1)$

(tj. $\xi_i \geq x_i$ i $\xi_i \leq x_{i+1}$)

$\xi_i = \sqrt{x_i \cdot x_{i+1}} \geq \sqrt{x_i \cdot x_i} = \sqrt{x_i^2} = x_i$

$\xi_i \leq \sqrt{x_{i+1} \cdot x_{i+1}} \leq \sqrt{x_{i+1}^2} = x_{i+1}$

$f(x) = \frac{1}{x^2} \Rightarrow f(\xi_i) = \frac{1}{\xi_i^2} = \frac{1}{x_i \cdot x_{i+1}} \quad (i = 0, 1, \dots, n-1)$

$$\int_a^b \frac{dx}{x^2} = \lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{1}{x_i \cdot x_{i+1}} \cdot h = \lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{x_{i+1}-x_i}{x_i \cdot x_{i+1}} =$$

$$= \lim_{n\to\infty} \sum_{i=0}^{n-1} \left(\frac{x_{i+1}}{x_i x_{i+1}} - \frac{x}{x_i \cdot x_{i+1}} \right) = \lim_{n\to\infty} \sum_{i=0}^{n-1} \left(\frac{1}{x_i} - \frac{1}{x_{i+1}} \right) =$$

$$= \lim_{n\to\infty} \left(\frac{1}{x_0} - \cancel{\frac{1}{x_1}} + \cancel{\frac{1}{x_1}} - \cancel{\frac{1}{x_2}} + \cancel{\frac{1}{x_2}} + \dots + \frac{1}{x_{n-1}} - \frac{1}{x_n} \right) = \frac{1}{a} - \frac{1}{b}$$

Provjera po N-L formuli:

$$\int_a^b \frac{dx}{x^2} = -\frac{1}{x} \Big|_a^b = -\left(\frac{1}{b} - \frac{1}{a} \right) = \frac{1}{a} - \frac{1}{b}$$

b) $\int_0^{\frac{\pi}{2}} \sin x \, dx$

Uputa: ekvidistantna podjela: $\xi_i = x_i$

c) $\int_{-1}^{1} (1+x)\, dx$:

Uputa: $\xi_i = \frac{x_i + x_{i+1}}{2}$

(2) $\int_{1,2}^{125,3} \frac{[x]}{x} dx = I$

najveće cijelo x : $[x]$, npr. $[3,6] = 3$

$x \in \mathbb{R}^+ \Rightarrow [\exists a \in \mathbb{N}] \quad a \le x < a+1 \quad \Rightarrow [x] = a$

$$I = \int_{1,2}^{2} \frac{[x]}{x} dx + \int_{2}^{3} \frac{[x]}{x} dx + \int_{3}^{4} \frac{[x]}{x} dx + \dots + \int_{124}^{125} \frac{[x]}{x} dx + \int_{125}^{125,3} \frac{[x]}{x} dx =$$

$$= \int_{1,2}^{2} \frac{1}{x} dx + \int_{2}^{3} \frac{2}{x} dx + \int_{3}^{4} \frac{3}{x} dx + \dots + \int_{124}^{125} \frac{124}{x} dx + \int_{125}^{125,3} \frac{125}{x} dx =$$

$$= \ln x\Big|_{1,2}^{2} + 2\ln x\Big|_{2}^{3} + 3\ln x\Big|_{3}^{4} + \dots + 124\ln x\Big|_{124}^{125} + 125\ln x\Big|_{125}^{125,3} =$$

$$= \ln 2 - \ln 1,2 + 2(\ln 3 - \ln 2) + 3(\ln 4 - \ln 3) + \dots + 124(\ln 125 - \ln 124) +$$

$$125(\ln 125,3 - \ln 125) = -\ln 1,2 - \ln 2 - \ln 3 - \ln 4 - \dots - \ln 124 - \ln 125 - \ln 125,3 =$$

$$= \ln \frac{125,3^{125}}{1,2\cdot 2\cdot 3\cdot 4\cdots 125} = \ln \frac{125,3^{125}}{1,2\cdot 125!}$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

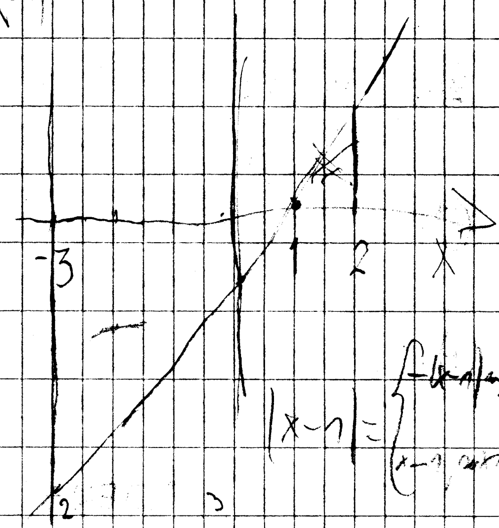
(3) Izračunati integrale

a) $\displaystyle\int_{1,3}^{150,2} [x]\, x^3\, dx$

b) $\displaystyle\int_{2,4}^{35,5} \frac{[x]}{x^4}\, dx$

$\Big\}$ za vježbu

$y = x - 1$



(4) $$\int_{-3}^{2} |x-1|\,dx = \int_{-3}^{1}|x-1|\,dx + \int_{1}^{2}|x-1|\,dx =$$

$$|x-1| = \begin{cases} -(x-1), & x<1 \\ x-1, & x\geq 1 \end{cases}$$

$$= -\int_{-3}^{1}(x-1)\,dx + \int_{1}^{2}(x-1)\,dx = -\int_{-3}^{1}x\,dx + \int_{-3}^{1}dx + \int_{1}^{2}x\,dx - \int_{1}^{3}dx =$$

$$= -\frac{x^2}{2}\Big|_{-3}^{1} + x\Big|_{-3}^{1} + \frac{x^2}{2}\Big|_{1}^{2} - x\Big|_{1}^{2} = -\frac{1}{2}(1-9) + 1-(-3) + \frac{1}{2}(4-1) - (2-1) = 4+4+\frac{3}{2} - 1 =$$

$$= 7 + \frac{3}{2} = \frac{17}{2}$$

(5*) $$\int_{-1}^{4}\sqrt{x^4 - 6x^3 + 9x^2}\,dx = \int_{-1}^{4}\sqrt{x^2(x^2-6x+9)}\,dx = \int_{-1}^{4}\sqrt{x^2(x-3)^2}\,dx =$$

$$= \int_{-1}^{4}|x(x-3)|\,dx = \int_{-1}^{0}|x(x-3)|\,dx + \int_{0}^{3}|x(x-3)|\,dx + \int_{3}^{4}|x(x-3)|\,dx =$$

$$= \int_{-1}^{0}x(x-3)\,dx - \int_{0}^{3}x(x-3)\,dx + \int_{3}^{4}x(x-3)\,dx =$$

$$= \int_{-1}^{0}(x^2-3x)\,dx - \int_{0}^{3}(x^2-3x)\,dx + \int_{3}^{4}(x^2-3x)\,dx$$

$$= \int_{-1}^{0} x^2 dx - 3\int_{-1}^{0} x\,dx - \int_{0}^{3} x^2 dx + 3\int_{0}^{3} x\,dx + \int_{3}^{4} x^2 dx + 3\int_{3}^{4} x\,dx =$$

$$= \frac{x^3}{3}\Big|_{-1}^{0} - 3\,\frac{x^2}{2}\Big|_{-1}^{0} - \frac{x^3}{3}\Big|_{0}^{3} + 3\,\frac{x^2}{2}\Big|_{0}^{3} + \frac{x^3}{3}\Big|_{3}^{4} - 3\,\frac{x^2}{2}\Big|_{3}^{4} =$$

$$= \frac{1}{3}(0+1) - \frac{3}{2}(0-1) - \frac{1}{\cancel{3}}\, 3^{\cancel{3}\,2} + \frac{3}{2}\cdot 3^2 + \frac{1}{3}(4^3-3^3) - \frac{3}{2}(4^2-3^2) =$$

$$= \frac{1}{3} + \frac{3}{2} - 9 + \frac{27}{2} + \frac{37}{3} - \frac{21}{2} = \frac{38}{3} + \frac{9}{2} - 9 = \frac{76+27-54}{6} = \frac{49}{6}$$

Za vježbu

(6) $\int_{0}^{1} \sqrt{8x^2-8x+2}\,dx$

(7) $\int_{-2}^{4} \left| |(x-2)| - 1 \right| dx$

SMJENA PROMJENLJIVE U ODREĐENOM INTEGRALU

$$\int_a^b f(g(x))\cdot g'(x)\,dx = \left| \begin{array}{ll} g(x) = t & x=a \Rightarrow t=g(a) \\ g'(x)dx = dt & x=b \Rightarrow t=g(b) \end{array} \right| =$$

$$= \int_{g(a)}^{g(b)} f(t)\,dt$$

(1) $$\int_1^e \frac{dx}{x(1+\ln^2 x)} = \left| \begin{array}{ll} \ln x = t & x=1 \Rightarrow t=\ln 1 = 0 \\ \frac{1}{x}dx = dt & x=e \Rightarrow t = \ln e = 1 \end{array} \right| = \int_0^1 \frac{dt}{1+t^2}$$

$$= \operatorname{arctg} t \Big|_0^1 = \operatorname{arctg} 1 - \operatorname{arctg} 0 = \frac{\pi}{4}$$

(2) $\int_0^4 x^3\sqrt{x^2+9}\,dx = \int_0^4 x^2\cdot x\sqrt{x^2+3^2}\,dx = \left| \begin{array}{ll} x^2=t^2-9 & x=0 \Rightarrow t^2=9\ t=3 \\ x^2+9=t^2 & x=4 \Rightarrow t^2=25 \\ 2xdx=2tdt & \\ xdx=tdt & \Rightarrow t=5 \end{array} \right| =$

$= \int_3^5 t(t^2-9)t\,dt = \int_3^5 (t^4-9t^2)\,dt = \int_3^5 t^4\,dt - 9\int_3^5 t^2\,dt =$

$= \frac{t^5}{5}\Big|_3^5 - \cancel{9}^3\,\frac{t^3}{3}\Big|_3^5 = \frac{1}{5}(5^5-3^5) - 3(5^3-3^3) = \frac{1}{5}(3125-243) - 3\cdot(125-27) = \dots$

(3)* $\int_0^3 x^2\sqrt{9-x^2}\,dx = \left| \begin{array}{ll} x=3\sin t & x=0 \Rightarrow \sin t=0 \Rightarrow t=0 \\ dx=3\cos t\,dt & x=3 \Rightarrow \sin t=1 \Rightarrow t=\frac{\pi}{2} \end{array} \right| =$

$= \int_0^{\frac{\pi}{2}} 9\sin^2 t\cdot\sqrt{9-9\sin^2 t}\cdot 3\cos t\,dt = 27\int_0^{\frac{\pi}{2}} \sin^2 t\cdot\sqrt{9(1-\sin^2 t)}\cdot\cos t\,dt =$

$= 81\int_0^{\frac{\pi}{2}} \sin^2 t\cos^2 t\,dt = 81\int_0^{\frac{\pi}{2}}\left(\frac{1}{2}\sin 2t\right)^2 dt = \frac{81}{4}\int_0^{\frac{\pi}{2}} \sin^2 2t\,dt =$

$= \frac{81}{4}\int_0^{\frac{\pi}{2}} \frac{1-\cos 4t}{2}\,dt = \frac{81}{8}\left(\int_0^{\frac{\pi}{2}} dt - \int_0^{\frac{\pi}{2}} \cos 4t\,dt\right) =$

$= \frac{81}{8}\left(t\Big|_0^{\frac{\pi}{2}} - \frac{1}{4}\underbrace{\sin 4t\Big|_0^{\frac{\pi}{2}}}_{0}\right) = \frac{81}{8}\cdot\frac{\pi}{2} = \frac{81\pi}{16}$

(4) $\int_1^{1,5} \frac{4x+3}{(x-2)^3}\,dx = \left| \begin{array}{ll} x-2=t & x=1 \Rightarrow t=-1 \\ dx=dt & x=1,5 \Rightarrow t=-0,5 \end{array} \right| =$

$= \int_{-1}^{-0,5} \frac{4(t+2)+3}{t^3}\,dt = \int_{-1}^{-0,5} \frac{4t+11}{t^3}\,dt = \int_{-1}^{0,5} \frac{4t}{t^3}\,dt + \int_{-1}^{-0,5} \frac{11}{t^3}\,dt =$

$= 4\int_{-1}^{-0,5} \frac{\cancel{t}}{t^{\cancel{3}\,2}}\,dt + 11\int_{-1}^{-0,5} \frac{1}{t^3}\,dt = 4\cdot\int_{-1}^{-0,5} t^{-2}\,dt + 11\int_{-1}^{-0,5} t^{-3}\,dt =$

$= 4\left(\frac{-1}{t}\right)\Big|_{-1}^{-0,5} + 11\,\frac{t^{-2}}{-2}\Big|_{-1}^{-0,5} = 4\left(\frac{-1}{0,5} - \frac{1}{(-1)}\right) - \frac{11}{2}\left(\frac{1}{0,5^2} - \frac{1}{1}\right) =$

$= -4(-2+1) - \frac{11}{2}(4-3) = 4 - \frac{33}{2} = -\frac{25}{2}$

Za vježbu:

(5) $\int_0^{\ln 5} \frac{\sqrt{e^x - 1}}{1 + 3 \cdot e^{-x}} dx$

(8) $\int_0^{11} \frac{(x-1)\,dx}{x + 2\sqrt{x-2}}$

(6) $\int_{\frac{1}{\pi}}^{\frac{2}{\pi}} \frac{1}{x^2} \sin\frac{1}{x}\, dx$

(9) $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{dx}{3 + \cos x}$ $\quad \left(\operatorname{tg}\frac{x}{2} = t \right)$

(7) $\int_{-2}^{2} x^2 \sqrt{4 - x^2}\, dx$

METODA PARCIJALNE INTEGRACIJE U ODREĐENOM INTEGRALU

$$\int_a^b u\, dv = uv \Big|_a^b - \int_a^b v\, du \qquad (u = u(x),\ v = v(x))$$

(1) $\int_0^{\frac{\pi}{4}} x \sin 2x\, dx \quad \left| \begin{matrix} u = x & dv = \sin 2x\, dx \\ du = dx & v = -\frac{1}{2}\cos 2x \end{matrix} \right| =$

$= -\frac{1}{2} x \cos 2x \Big|_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \left(-\frac{1}{2}\right) \cos 2x\, dx = -\frac{1}{2}\frac{\pi}{4}\cos\frac{2\pi}{4} + \frac{1}{2}\int_0^{\frac{\pi}{4}} \cos 2x\, dx =$

$= \frac{1}{2} \cdot \frac{1}{2} \sin 2x \Big|_0^{\frac{\pi}{4}} = \frac{1}{4} \sin\frac{2\pi}{4} = \frac{1}{4}$

(2)* $\int_1^4 |\ln x - 1|\, dx = -\int_1^e (\ln x - 1)\, dx + \int_e^4 (\ln x - 1)\, dx =$

$= -\int_1^e \ln x\, dx + \int_1^e dx + \int_e^4 \ln x\, dx - \int_e^4 dx = \left| \begin{matrix} \ln x = u & v = \int dx \\ \frac{1}{x} dx = du & v = x \end{matrix} \right| =$

$$= -\left(\ln x \cdot x \Big|_1^e - \int_1^e x \cdot \frac{1}{x}\,dx\right) + x\Big|_1^e + \ln x \cdot x\Big|_e^4 - \int_e^4 x \cdot \frac{1}{x}\,dx - x\Big|_e^4 =$$

$$= -\left(e\ln e - x\Big|_1^e\right) + e - 1 + 4\ln 4 - e\ln e - x\Big|_e^4 - x\Big|_e^4 =$$

$$= -e + e - 1 + e - 1 + 4\ln 4 - e - 2(4-e) = 4\ln 4 - 10 + 2e$$

(3)

$$I = \underbrace{\int_{-\frac{1}{2}}^{\frac{1}{2}} x^2 \ln \frac{1-x}{1+x}}_{f(x)}\,dx$$

$$f(-x) = (-x)^2 \ln \frac{1+x}{1-x} = x^2 \ln\left(\frac{1-x}{1+x}\right)^{-1} = -x^2 \ln \frac{1-x}{1+x} = -f(x)$$

$f(-x) = -f(x) \Rightarrow$ f-ija je neparna $\Rightarrow I = 0$

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(4)

$$I = \int_{-1}^{1} \frac{x^5 - 2x^3 + 4x + 2}{\sqrt{x^2+1}}\,dx = \underbrace{\int_{-1}^{1} \frac{x^5}{\sqrt{x^2+1}}\,dx}_{0} + \underbrace{\int_{-1}^{1} \frac{2x^3}{\sqrt{x^2+1}}\,dx}_{0} + \underbrace{\int_{-1}^{1} \frac{4x}{\sqrt{x^2+1}}}_{0} + 2\int_{-1}^{1} \frac{dx}{\sqrt{x^2+1}}$$

korijen

$$I = 2\int_{-1}^{1} \frac{dx}{\sqrt{x^2+1}} = 2 \cdot 2 \cdot \int_0^1 \frac{dx}{\sqrt{x^2+1}} = 4 \ln\left|x + \sqrt{x^2+1}\right| \Big|_0^1 =$$

$$= 4\left(\ln(1+\sqrt{2}) - \underbrace{\ln 1}_{0}\right) = 4\ln(1+\sqrt{2})$$

(5)

$$I = \int_{-1}^{1} \underbrace{x \cdot \operatorname{arctg} x}_{f(x)}\,dx =$$

parna

$$f(-x) = -x \cdot \operatorname{arctg}(-x) = -(-x)\operatorname{arctg} x = x \operatorname{arctg} x = f(x)$$

$$I = 2\cdot\int_0^1 x \operatorname{arctg} x\,dx = \left| \begin{matrix} u = \operatorname{arctg}x & dv = x\,dx \\ du = \frac{1}{1+x^2}\,dx & v = \frac{x^2}{2}\end{matrix}\right| =$$

$$= \cancel{2}\left(\frac{x^2}{\cancel{2}}\operatorname{arctg}x\Big|_0^1 - \int_0^1 \frac{x^2}{\cancel{2}}\cdot\frac{1}{1+x^2}\,dx\right) = \operatorname{arctg}1 - \int_0^1 \frac{x^2+1-1}{1+x^2}\,dx =$$

$$= \frac{\pi}{4} - \left(\int_0^1 \frac{x^2+1}{1+x^2}\,dx - \int_0^1\frac{1}{1+x^2}\,dx\right) = \frac{\pi}{4} - \left(x\Big|_0^1 - \operatorname{arctg}x\Big|_0^1\right)$$

$$= \frac{\pi}{4} - (1 - \operatorname{arctg}1) = \frac{\pi}{4} - 1 + \frac{\pi}{4} = \frac{2\pi}{4} - 1 = \frac{\pi}{2} - 1 = \frac{\pi-2}{2}$$

za vježbu:

6) $I = \int_{\frac{1}{e}}^{e} |\ln x|\,dx$

9) $\int_0^e \sin(\ln x)\,dx$

7) $\int_1^2 \ln\left(1+\frac{1}{x}\right)dx$

10) $\int_0^{\frac{\pi}{11}} (x+2\pi)\cos 11x\,dx$

8) $\int_0^{\frac{5\pi}{3}} x\,|\sin x|\,dx$

11) $\int_0^1 \arcsin\frac{x}{2}\,dx$